

Comments on the 2nd order bootstrap relation

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Abstract. The 2nd order bootstrap relation is discussed in view of the recent critics by F.Fadin, R.Fiore y A.Papa. It is shown that the strong bootstrap condition and the anzatz to solve it used in our earlier paper are valid at least for the quark part of the next-to-leading contribution.

1 Introduction.

The bootstrap relation plays a key role in the derivation of the BFKL equation up to the 2nd order in α_s . It guarantees that production amplitudes with the gluon quantum number in their t channels used for the construction of the absorptive part are indeed given only by a single reggeized gluon exchange and do not contain admixture from two or more reggeized gluon exchanges. The bootstrap relation is known to be satisfied in the lowest order in the coupling constant. Recently the 2nd order bootstrap relation was discussed in [1]. In a stronger form it was used in [2] to obtain the potential in the t -channel with the gluon colour quantum number. This approach was lately criticized in [3], where it was claimed that the strong bootstrap condition used in [2] was not fulfilled and the anzatz used to solve it was incorrect. In this note we demonstrate that these objections are totally unfounded. They are a result of a misinterpretation of the potential used in [2], which is a different quantity as compared to the kernel used in [1,3]. We also derive the 2nd order bootstrap relation of [1] in a simpler way and comment on its implication for particle-reggeon scattering amplitudes.

2 Formalism

To introduce the bootstrap relation in a general form it is convenient to use an operator formalism for the (non-forward) two gluon equation. Let the two gluons with momenta $q_{1,2}$ be described by a wave function $\Psi(q_1, q_2)$. The total momentum $q = q_1 + q_2$ will always be conserved, so that in future $q_2 = q - q_1$ and the dependence on q and thus q_2 will be suppressed. To facilitate comparison with [1,3] we introduce a metric in which the scalar product of two wave functions is given by

$$\langle \Psi_1 | \Psi_2 \rangle \equiv \int d^{D-2} q_1 \Psi_1^*(q_1) \Psi_2(q_1), \quad (1)$$

where $D = 4 + 2\epsilon$, $\epsilon \rightarrow 0$, is the dimension used to regularize all the following expressions in the infrared region.

In this metric, up to order α_s^3 , the absorptive part of the scattering amplitude coming from the two-reggeized-gluon exchange is given by

$$\mathcal{A}_j = \langle \Phi_p | G(E) | \Phi_t \rangle \quad (2)$$

Here the functions $\Phi_{p,t}$ are the so called impact factors which represent the coupling of the external particles (t from the target and p from the projectile) to the two exchanged gluons. They are low energy particle-reggeon scattering amplitudes. They have order $\alpha_s = g^2/4\pi$. The Green function $G(E)$ with $E = 1 - j$ is defined as

$$G(E) = (H - E)^{-1} \quad (3)$$

where H is the two gluon (Hermithean) Hamiltonian. Its explicit expression in the momentum representation is

$$\langle q_1 | H | q'_1 \rangle = -\delta^{D-2}(q_1 - q'_1)(\omega(q_1) + \omega(q_2)) - V^{(R)}(q_1, q'_1) \quad (4)$$

where $1 + \omega(q_{1(2)})$ is the Regge trajectory of the first (second) gluon and $V^{(R)}$ represents the gluonic interaction for the t -channel with the colour quantum number R . In the lowest (1st) order in α_s one has [4]

$$\omega^{(1)}(q) = -aq^2 \langle \chi | \chi \rangle \quad (5)$$

where

$$\chi(q_1) = 1/\sqrt{q_1^2 q_2^2}, \quad (6)$$

$$a = \frac{g^2 N}{2(2\pi)^{D-1}} \quad (7)$$

and N is the number of colours. As to the gluonic interaction, in the 1st order

$$V^{(R)}(q_1, q'_1) = \frac{ac_R}{\sqrt{q_1^2 q_2^2 q_1'^2 q_2'^2}} \left(\frac{q_1^2 q_2'^2 + q_2^2 q_1'^2}{k^2} - q^2 \right) \quad (8)$$

with $k = q_1 - q'_1$ (the 1st denominator appears due to our metric (1)) The coefficient c_R depends on the colour quantum number of the t -channel. For the vacuum channel ($R = v$) $c_v = 2$. For the channel with the gluon colour quantum number ($R = g$) $c_g = 1$.

In any colour channel the Green function $G(E)$ can be represented via the solutions of the homogeneous Schroedinger equation

$$H\Psi_n = E_n\Psi_n. \quad (9)$$

These solution may be taken orthonormalized

$$\langle \Psi_m | \Psi_n \rangle = \delta_{mn} \quad (10)$$

(of course index n may be continuous). They are assumed to form a complete set

$$\sum_n |\Psi_n\rangle \langle \Psi_n| = 1. \quad (11)$$

In our metric the last equation means in the momentum space

$$\sum_n \Psi_n(q_1) \Psi_n^*(q'_1) = \delta^{D-2}(q_1 - q'_1). \quad (12)$$

In terms of the eigenfunctions Ψ_n

$$G(E) = \sum_n \frac{|\Psi_n\rangle\langle\Psi_n|}{E_n - E} \quad (13)$$

and as a consequence the absorptive part \mathcal{A}_j is expressed as

$$\mathcal{A}_j = \sum_n \frac{\langle\Phi_p|\Psi_n\rangle\langle\Psi_n|\Phi_t\rangle}{E_n - E}. \quad (14)$$

Note that sometimes it is convenient to consider not the absorptive part of the amplitude for real particles but amplitudes for particle-reggeon scattering. One can evidently define two such amplitudes $\Psi_{p,t}$ depending on which real particle is taken, from the projectile or target. Formally we define

$$\Psi_t(E) = G(E)|\Phi_t\rangle. \quad (15)$$

It satisfies an inhomogeneous Schroedinger equation

$$(H - E)\Psi_t(E) = \Phi_t. \quad (16)$$

The absorptive part can then be expressed as

$$\mathcal{A}_j = \langle\Phi_p|\Psi_t(E)\rangle. \quad (17)$$

In terms of the eigenfunctions Ψ_n one can express

$$\Psi_t(E) = \sum_n \frac{\Psi_n\langle\Psi_n|\Phi_t\rangle}{E_n - E}. \quad (18)$$

The other function $\Psi_p(E)$ can be introduced in a similar manner.

3 The bootstrap conditions

Now we concentrate on the t -channel with a colour quantum number of the gluon ($R = g$). The ideology of the BFKL equation is based on the idea that physical amplitudes in this channel have the asymptotical behaviour corresponding to an exchange of a single reggeized gluon. The important requirement is thus that no contribution should come from the exchange of two or more reggeized gluons (in the leading and next-to-leading orders). This requirement seems to be automatically fulfilled due to the Gribov's rule of signature conservation. Indeed the two-gluon exchange should have an overall positive signature and thus cannot contribute to the amplitude with the t channel corresponding to the gluon colour quantum number, which has a negative signature. The bootstrap relation is just a technical tool to implement this requirement.

In order that the amplitude should have the asymptotic behaviour corresponding to a single reggeized gluon exchange without admixture of any other terms, its absorptive part, considered as a function of the complex angular momentum j , should have a simple pole at $j = 1 + \omega(q)$ and no other singularities. From the representation (13) we conclude that

First, the spectrum of H in the considered t channel should contain an eigenvalue $E_0 = -\omega(q)$ with the corresponding eigenfunction Ψ_0

$$(H + \omega(q))\Psi_0 = 0 \quad (19)$$

Second, in the sum over n only the contribution of this eigenfunction Ψ_0 should be present. All other terms should be equal to zero. Strictly speaking, this means that both impact factors $\Phi_{p,t}$ should be orthogonal to all other eigenfunctions Ψ_n , $n > 0$. Due to the completeness of the whole set it means that both Φ_p and Φ_t should coincide with Ψ_0 , up to a normalization factor.

This second condition looks to be very stringent. Note that it should be fulfilled for *any* projectile and/or target. So, literally understood, it means that coupling of any particle to a pair of reggeons is essentially the same function of the reggeonic momenta. As we shall see below, in fact, this condition is far weaker, due to the fact that all our arguments are limited to the two first orders of the perturbation theory.

There is finally a third bootstrap condition which is a relation for the impact factor. With only the state Ψ_0 contributing we get from (13)

$$\mathcal{A}_j = \frac{\langle \Phi_p | \Psi_0 \rangle \langle \Psi_0 | \Phi_t \rangle}{j - 1 - \omega(q)} \quad (20)$$

Integrating this over j with a proper signature factor we find the corresponding amplitude as a function of s

$$\mathcal{A}(s) = -s \left(\frac{q^2}{s_0} \right)^{\omega(q)} \frac{\langle \Phi_p | \Psi_0 \rangle \langle \Psi_0 | \Phi_t \rangle}{\sin \pi \omega(q)} \left[\left(\frac{-s}{q^2} \right)^{\omega(q)} + \left(\frac{s}{q^2} \right)^{\omega(q)} \right] \quad (21)$$

This should be compared with the standard form of the contribution of a Reggeized gluon, with a scale given by q^2 ,

$$\mathcal{A}(s) = -\Gamma_p(q) \Gamma_t(q) \frac{s}{q^2} \left[\left(\frac{-s}{q^2} \right)^{\omega(q)} + \left(\frac{s}{q^2} \right)^{\omega(q)} \right] \quad (22)$$

Here Γ 's represent the coupling of the projectile or target particles to the exchanged reggeized gluon ("particle-particle-Reggeon vertices", in the terminology of [1]). Comparing (21) and (22) we find a relation which should be satisfied both for the projectile and target

$$\Gamma(q) = q \left(\frac{q^2}{s_0} \right)^{\omega(q)/2} \frac{\langle \Phi | \Psi_0 \rangle}{\sqrt{|\sin \pi \omega(q)|}} \quad (23)$$

In fact the impact factor Φ itself involves the vertex Γ (in the lowest order it is completely expressed through it). So Eq. (23) is a non-trivial condition to be satisfied for the coupling of any particle to the reggeized gluon. It is the third bootstrap condition.

4 Leading and next-to-leading orders

It is well known that the bootstrap conditions described in the preceding section are fulfilled in the leading order (LO) [5]. Indeed the homogeneous LO Schroedinger equation has an eigenvalue $E_0^{(1)} = -\omega^{(1)}(q)$ with the corresponding eigenfunction $\Psi_0^{(0)} = c\chi$ (Eq. (6), $c = 1/\sqrt{\langle \chi | \chi \rangle}$ is the normalization factor):

$$H^{(1)}\chi = -\omega^{(1)}(q)\chi \quad (24)$$

(upper indeces always mark the order in α_s). One easily checks that (24) is true with the explicit expressions (4)–(8) for the Hamiltonian and χ . Due to the trivial form of the latter Eq. (24) is in fact a simple relation between the gluon trajectory and the gluonic interaction integrated over one of its momenta.

It can also be trivially shown that in the lowest order the impact factors for any external particle reduce to χ as required. Indeed (see[1]) in the LO for any particle

$$\Phi = -i\frac{\sqrt{N}}{2}\Gamma\chi. \quad (25)$$

From this one gets

$$\langle\Psi_0^{(0)}|\Phi\rangle = -i\frac{\sqrt{N}}{2}\Gamma\sqrt{\langle\chi|\chi\rangle} = \frac{\sqrt{\pi\omega^{(1)}}}{q}\Gamma \quad (26)$$

where we have used (5). Putting this in (23) we find that also the third bootstrap condition is satisfied in the LO, when we can neglect the power factor and take $\sin\pi\omega \simeq \pi\omega$

Now we pass to the next-to-leading order (NLO). Again we have to fulfil three different requirements. First, the spectrum of the Hamiltonian should include the gluon trajectory also in the NLO, Eq. (19) We present the corresponding eigenfunction as

$$\Psi_0 = c\chi + \Psi_0^{(1)} \quad (27)$$

where χ is the known LO eigenfunction and the second term is the (unknown) NLO contribution. Separating perturbation orders in the Hamiltonian and in the eigenvalue $E_0 = -\omega(q)$ and taking the NLO we get an equation for $\Psi_0^{(1)}$

$$(H^{(1)} + \omega^{(1)}(q))\Psi_0^{(1)} = -c(H^{(2)} + \omega^{(2)}(q))\chi \quad (28)$$

For this equation to be soluble, the inhomogeneous term should be orthogonal to the solution of the corresponding homogeneous equation, that is, to χ , recall Eq. (24). So we get a condition

$$\langle\chi|H^{(2)} + \omega^{(2)}(q)|\chi\rangle = 0 \quad (29)$$

With χ a known function, this equation is in fact a relation between the gluonic NLO interaction in the colour channel $R = g$ and the NLO Regge trajectory, integrated over the initial and final gluon relative momenta. This relation was first obtained in [1] from different arguments. Note that in contrast to the Eq. (19), valid for any q and q_1 , relation (29) is only an identity in q . So its contents is much weaker than that of (19). For this reason we shall call it a weak bootstrap condition, leaving the definition of the strong bootstrap condition for the equations (24) in the LO and (19) in the LO and NLO.

Now we come to the second condition, which is that also in the NLO there should be no contribution to the absorptive part other than from the gluonic Regge pole. However this condition is trivially satisfied in the NLO. Indeed matrix elements

$$\langle\Phi_{p,t}|\Psi_n\rangle, \quad n \neq 0$$

with eigenfunctions different from Ψ_0 are all zero in the lowest order. So they have at least order α_s . The representation (13) contains products of two such matrix elements. So the part of (13) which comes from the states other than Ψ_0 is at least two orders smaller than the leading term (of the order α_s^2). Therefore it does not contribute in the NLO at all. Thus in the NLO, irrespective of the form of the impact factors, there is no contribution to the absorptive part of the amplitude from states other than a single reggeized gluon.

It is remarkable that this is only true for the absorptive part of the scattering amplitude for real particles. If one considers the particle- reggeon amplitudes instead, then admixture of other intermediate states, apart from a single reggeized gluon, seems possible. Indeed take $\Psi_t(E)$ as an example. One immediately sees from Eq. (18) that terms with $n \neq 0$ will generally give a nonzero contribution in the NLO unless the impact factor coincides with the eigenfunction Ψ_0 also in the the NLO (which seems improbable for every possible external particle). This contribution disappears when one takes the product in (17) due to orthogonality of the eigenfunction with $n = 0$ to all others. However in the particle-reggeon amplitude $\Psi_t(E)$ itself the NLO contribution from higher gluonic states may be present.

This circumstance makes one think about the presence of such states in the reggeon-reggeon amplitudes which enter the unitarity relation for the absorptive part of the initial amplitude. Should such states be present, it would invalidate the derivation of the BFKL equation in the NLO.

The form which takes the third bootstrap condition (23) in the NLO was obtained in [1]. As far as we know it has not been checked for any particle so far.

5 Solution of the strong bootstrap condition

In this section we are going to demonstrate that the ansatz proposed in [6] and used in [2] actually solves the strong bootstrap relation (19) in the LO and NLO orders. The ansatz introduces a single function $\eta(q)$ through each both the potential and the gluon Regge trajectory are expressed. In [2] we used an unsymmetric potential W defined as follows

$$W(q_1, q'_1) = \left(\frac{\eta(q_1)}{\eta(q'_1)} + \frac{\eta(q_2)}{\eta(q'_2)} \right) \frac{1}{\eta(k)} - \frac{\eta(q)}{\eta(q'_1)\eta(q'_2)} \quad (30)$$

The trajectory is expressed via $\eta(q)$ as

$$\omega(q) = - \int d^{D-2} q_1 \frac{\eta(q)}{\eta(q_1)\eta(q_2)} \quad (31)$$

This ansatz guarantees fulfilment of the bootstrap relation in the form

$$\int d^{D-2} q'_1 W(q_1, q'_1) = \omega(q) - \omega(q_1) - \omega(q_2) \quad (32)$$

In the LO one has

$$\eta^{(0)}(q) = q^2/a \quad (33)$$

(see Eq. (7)).

To pass to the symmetric potential V one has first separate from W the integration measure factor $1/(\eta(q'_1)\eta(q'_2))$ and then symmetrize substituting this factor for $1/\sqrt{\eta(q_1)\eta(q_2)\eta(q'_1)\eta(q'_2)}$. This gives the desired relation

$$W(q_1, q'_1) = \sqrt{\frac{\eta(q_1)\eta(q_2)}{\eta(q'_1)\eta(q'_2)}} V(q_1, q'_1) \quad (34)$$

The bootstrap relation (32) then transforms into

$$\int d^{D-2} q'_1 V(q_1, q'_1) \frac{1}{\sqrt{\eta(q'_1)\eta(q'_2)}} = (\omega(q) - \omega(q_1) - \omega(q_2)) \frac{1}{\sqrt{\eta(q_1)\eta(q_2)}} \quad (35)$$

This is nothing but the equation (19) for the gluon Regge pole solved by

$$\Psi_0(q_1) = c \frac{1}{\sqrt{\eta(q_1)\eta(q_2)}} \quad (36)$$

So our ansatz just solves the strong bootstrap relation.

To compare our results with [3] we have finally to relate our symmetric potential V to the irreducible kernel K_r used in [1,3]. The latter is defined in respect to the metric with a factor $q_1^2 q_2^2$ in the denominator. Therefore identification with V requires passing to our metric. We then find

$$V(q_1, q'_1) = \frac{K_r(q_1, q'_1)}{\sqrt{q_1^2 q_2^2 q_1'^2 q_2'^2}} \quad (37)$$

Combining (34) and (37) we find the final relation of the irreducible kernel K_r of [1,3] and the unsymmetric potential W of [2]

$$K_r(q_1, q'_1) = \sqrt{q_1^2 q_2^2 q_1'^2 q_2'^2} \sqrt{\frac{\eta(q'_1)\eta(q'_2)}{\eta(q_1)\eta(q_2)}} W(q_1, q'_1) \quad (38)$$

This relation has not, in all probability, been taken into account in [3] when discussing our results. In the following section we shall show that with the form of $\eta(q)$ matched to the quark contribution to the gluon Regge trajectory one obtains the correct form of the quark contribution to the kernel K_r .

6 The quark contribution to the trajectory and potential

We are going to show that, first, the quark contribution to the gluon Regge trajectory has the form implied by our ansatz (31). We shall determine the part of $\eta(q)$ coming from this contribution. Then using (30) and (38)) we shall find the corresponding part of the irreducible kernel K_r and compare it with the results of [3].

The part of the gluon trajectory which comes from quark is given by the expression [3]

$$\omega_Q^{(2)} = 2bq^2 \int \frac{d^{D-2}q_1}{q_1^2 q_2^2} (q^{2\epsilon} - q_1^{2\epsilon} - q_2^{2\epsilon}), \quad (39)$$

where

$$b = \frac{g^4 N N_F \Gamma(1-\epsilon) \Gamma^2(2+\epsilon)}{(2\pi)^{D-1} (4\pi)^{2+\epsilon} \epsilon \Gamma(4+2\epsilon)}. \quad (40)$$

On the other hand, presenting

$$\eta(q) = \eta^{(0)}(q)(1 + \xi(q)) \quad (41)$$

we have in the NLO from (31)

$$\omega^{(2)}(q) = -aq^2 \int \frac{d^{D-2}q_1}{q_1^2 q_2^2} (\xi(q) - \xi(q_1) - \xi(q_2)) \quad (42)$$

As we observe, the form of (39) follows this pattern. Comparing (39) and (42) we identify

$$\xi_Q(q) = -\frac{2b}{a} q^{2\epsilon} \quad (43)$$

Now we pass to the irreducible kernel K_r . From (30) and (38) we express it via $\xi(q)$ as

$$K_r^{(2)}(q_1, q'_1) = \frac{1}{2}a \sqrt{q_1^2 q_2^2 q_1'^2 q_2'^2} \left[\frac{1}{k^2} \sqrt{\frac{q_1^2 q_2'^2}{q_1'^2 q_2^2}} (\xi(q_1) + \xi(q'_2) - \xi(q'_1) - \xi(q_2) - 2\xi(k)) + \right.$$

$$\frac{1}{k^2} \sqrt{\frac{q_2^2 q_1'^2}{q_2'^2 q_1^2}} (\xi(q_2) + \xi(q_1') - \xi(q_1) - \xi(q_2') - 2\xi(k)) - \frac{q^2}{\sqrt{q_1^2 q_2^2 q_1'^2 q_2'^2}} (2\xi(q) - \xi(q_1) - \xi(q_2) - \xi(q_1') - \xi(q_2')) \quad (44)$$

Putting here the found $\xi(q)$, Eq. (43), we obtain the quark contribution

$$K_r^Q(q_1, q_1') = -b \left[\frac{q_1^2 q_2'^2}{k^2} (q_1^{2\epsilon} + q_2'^{2\epsilon} - q_2^{2\epsilon} - q_1'^{2\epsilon} - 2k^{2\epsilon}) + \frac{q_2^2 q_1'^2}{k^2} (q_2^{2\epsilon} + q_1'^{2\epsilon} - q_1^{2\epsilon} - q_2'^{2\epsilon} - 2k^{2\epsilon}) - q^2 (2q^{2\epsilon} - q_1^{2\epsilon} - q_2^{2\epsilon} - q_1'^{2\epsilon} - q_2'^{2\epsilon}) \right] \quad (45)$$

Comparing this expression with the one found in [3] (Eq. (47) of that paper) we observe that they are identical. This means that our bootstrap condition and the ansatz to solve it are valid at least for the quark contribution in the NLO.

7 Conclusions

Comparing our bootstrap results [2] for the gluonic interaction in the gluon colour channel with the direct calculations of its quark part in [3] we find a complete agreement. This leaves a certain dose of optimism as to the total potential calculated in [2] being the correct one.

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